Rolle's Theorem :

If a function $f:[a,b]\to R$ is

i) Continuous on $[a,b]$

ii) derivable on (a,b) and

iii) $f(a)=f(b)$, then

There exists atleast one real number $c\in(a,b)$ so that $f'(c)=0$

Proof : *Given* function $f:[a,b]\to R$ is

i) Continuous on $[a,b]$

ii) derivable on (a,b) and

iii) $f(a)=f(b)$, then

Now i) $\Rightarrow$ function f is Continuous on $[a,b]$

$\Rightarrow$ f is bounded and attain its bounds on $[a,b]$

$\therefore$ *Let* M & m be the bounds i.e Supremum & Infimum of f on $[a,b]$.

Then there exists $c,d\in[a,b]$ so that $f(c)=M$ & $f(d)=m$

Here there are two possibilities.

They are either $M=m$ or $M\neq m$

$Case\ 1)\ When\ M = m$

$$\text{Then } f(x) = M = m, \forall x \in [a,b]$$

$$\Rightarrow f \text{ is a constant function on } [a,b]$$

$$\Rightarrow f'(x) = 0, \forall x \in [a,b]$$

$$\therefore \textit{The} \text{ theorem is true.}$$

$Case\ 2)\ When\ M \neq m$

$$\text{Now } iii) \Rightarrow f(a) = f(b)$$

$$\Rightarrow \textit{Atleast} \text{ one of the values of M or m will be different from the equal values f(a) \& f(b)}$$

$$\text{Suppose M} = \text{f(c) is different from f(a) \& f(b)}$$

$$\text{i.e } f(c) \neq f(a) \ \ \& \ f(c) \neq f(b)$$

$$\Rightarrow c \neq a \ \ \& \ c \neq b$$

$$\Rightarrow c \notin (a,b)$$

$$\text{Also } ii) \Rightarrow f \text{ is derivable on (a,b)}$$

$$\Rightarrow f \text{ is derivable on at c , as c} \in \text{(a,b)}$$

$$f'(c-) = f'(c+) \text{ -----(1)}$$

Cleasrly, *we* know that $f(c)[=M]$ is the supremum of f on $[a,b]$

$\Rightarrow f(x) \leq f(c), \forall\, x \in [a,b]$

In particular $f(c-h) \leq f(c)$ & $f(c+h) \leq f(c)$ ----(2)

$$Now\ f'(c-) = \underset{x \to c-}{Lt} \frac{f(x)-f(c)}{x-c}$$

$$= \underset{h \to 0}{Lt} \frac{f(c-h)-f(c)}{-h}$$

≥ 0, from (2)

$$\text{Similarly } f'(c+) = \underset{x \to c+}{Lt} \frac{f(x)-f(c)}{x-c}$$

$$= \underset{h \to 0}{Lt} \frac{f(c+h)-f(c)}{h}$$

≤ 0, from (2)

But (1) $\Rightarrow f'(c-) = f'(c+)$

i.e both the limits must be equal.

$\therefore$ *This* happened only when $f'(c-)=0$ & $f'(c+)=0$

$\therefore\ f'(c)=0$, *where* $c \in (a,b)$

i.e $\exists\, c \in (a,b) \ni f'(c)=0$

Similarly $\exists\, d \in (a,b) \ni f'(d)=0$

Hence the Rolle's theorem

$Lagrange's$ Mean Value Theorem :

If a function $f:[a,b]\to R$ is

(*i*) Continuous on $[a,b]$ and

(*ii*) derivable on (a,b), then

there exists atleast one real number $c \in (a,b)$ so that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Pr*oof* : *Given* function $f:[a,b]\to R$ is

(*i*) Continuous on $[a,b]$ and

(*ii*) derivable on (a,b)

Define a function $\phi:[a,b]\to R$ such that $\phi(x) = f(x) + Kx$ -----(1)

where K is a constant to be determined so that $\phi(a) = \phi(b)$ ----(2)

$$Now(2) \Rightarrow \phi(a) = \phi(b)$$

$$\Rightarrow f(a) + Ka = f(b) + Kb$$

$$\Rightarrow K(a-b) = f(b) - f(a)$$

$$\Rightarrow -K = \frac{f(b)-f(a)}{b-a} \quad \text{----}(3)$$

*W*e know that f is continous on $[a,b]$ from (i) and K, being a constant function, is continuous on R ie on $[a,b]$, we have

$f(x) + Kx$ is continuous on $[a,b]$

$\Rightarrow \phi(x)$ is continuous on $[a,b]$, from (1)

lll^{ly} we know that f is derivable on (a,b) from (i) and K, being a constant function, is derivable on R ie on (a,b), we have

$f(x)+Kx$ is derivable on (a,b)

$\Rightarrow \phi(x)$ is continuous on (a,b), from (1)

$\therefore$ The function $\phi:[a,b]\rightarrow R$ is

i) Continuous on [a,b]

ii) derivable on (a,b) and

iii) $\phi(a)=\phi(b)$

$\therefore \phi$ satisfies all the conditions of Rolle's theorem.

$\therefore$ By Rolle's theorem $\exists\, c\in(a,b)$ so that $\phi'(c)=0$ - - - (4)

$$Now\ (1)\Rightarrow \varphi(x)=f(x)+Kx$$

$$\Rightarrow \varphi'(x)=f'(x)+K$$

$$\Rightarrow \varphi'(c)=f'(c)+K$$

$$\Rightarrow 0=f'(c)+K,\ from(4)$$

$$\Rightarrow f'(c)=-K$$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a},\ \text{from (3)}$$

$\therefore$ there exists atleast one real number $c\in(a,b)$ so that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Hence the theorem

$Cauchy's$ Mean value Theorem :

If two functions $f:[a,b] \to R, g:[a,b] \to R$ are

(*i*) Continuous on $[a,b]$

(*ii*) derivable on (a, b) and

(iii) $g^{'}(x) \neq 0, \forall\, x \in (a,b)$, then

there exists atleast one real number $c \in (a,b)$ so that $\frac{f^{'}(c)}{g^{'}(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

Pr*oof* : *Given* two functions $f:[a,b] \to R, g:[a,b] \to R$ are

(*i*) Continuous on $[a,b]$

(*ii*) derivable on (a, b) and

(iii) $g^{'}(x) \neq 0, \forall\, x \in (a,b)$

Define a function $\phi : [a,b] \to R$ such that $\phi(x) = f(x) + Kg(x)$ -----(1)

where K is a constant to be determined so that $\phi(a) = \phi(b)$ ----(2)

$Now (2) \Rightarrow \phi(a) = \phi(b)$

$\Rightarrow f(a) + Kg(a) = f(b) + Kg(b)$

$\Rightarrow K[g(a) - g(b)] = f(b) - f(a)$

$\Rightarrow -K = \frac{f(b) - f(a)}{g(b) - g(a)}$ ----(3)

Because $g(a) \neq g(b)$, Otherwise,

By Rolle's th. $\exists\, c \in (a,b)$ so that $g^{'}(c) = 0$

which is a contradiction to (iii)

We know that f,g are continous on [a, b] from (i) and K, being a constant function, is continuous on R ie on [a, b], we have

$f(x) + Kg(x)$ is continuous on [a, b]

$\Rightarrow \phi(x)$ is continuous on [a, b], from (1)

lll^{ly} we know that f,g are derivable on (a, b) from (i) and K, being a constant function, is derivable on R ie on (a, b), we have

$$f(x)+Kg(x) \text{ is derivable on } (a,b)$$

$$\Rightarrow \phi(x) \text{ is derivable on } (a,b), \text{ from } (1)$$

$\therefore$ The function $\phi:[a,b]\to R$ is

i) Continuous on $[a,b]$

ii) derivable on (a,b) and

iii) $\phi(a)=\phi(b)$

$\therefore \phi$ satisfies all the conditions of Rolle's theorem.

$\therefore$ By Rolle's theorem $\exists\, c\in(a,b)$ so that $\phi'(c)=0$ - - - (4)

$$Now\ (1)\Rightarrow \varphi(x)=f(x)+Kg(x)$$

$$\Rightarrow \varphi'(x)=f'(x)+Kg'(x)$$

$$\Rightarrow \varphi'(c)=f'(c)+Kg'(c)$$

$$\Rightarrow 0=f'(c)+Kg'(c),\ from (4)$$

$$\Rightarrow \frac{f'(c)}{g'(c)}=-K$$

$$\Rightarrow \frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}, \text{ from } (3)$$

$\therefore$ there exists atleast one real number $c\in(a,b)$ so that $\dfrac{f'(c)}{g'(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

Hence the theorem

$\text{Prob}: \textit{Verify}$ Rolle's theorem for the function $f(x)=(x-a)^m(x-b)^n, m,n \in Z^+$, *in* the interval [a, b]

$\textit{Sol}: \textit{Given}$ function is $f(x)=(x-a)^m(x-b)^n, m,n \in Z^+$, *in* the interval [a, b]

Clearly $f(x)$, *being* a polynomial function, is continuous on R i.e on [a, b]

$$\textit{Also } f'(x)=(x-a)^m[n(x-b)^{n-1}]+(x-b)^n[m(x-a)^{m-1}]$$
$$=(x-a)^{m-1}(x-b)^{n-1}[n(x-a)+m(x-b)]$$

$$\Rightarrow f'(x)=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)] \quad \text{----(1)}$$

clearly this exists

$\therefore f'(x)$ exists i.e f is derivable on (a, b)

Also $f(a)=0$ & $f(b)=0$ so that $f(a)=f(b)$

$\therefore f$ satisfies all the conditions of Rolle's theorem.

$\therefore$ By Rolle's theorem $\exists\, c \in (a,b)$ so that $f'(c)=0$

$$\Rightarrow f'(c)=(c-a)^{m-1}(c-b)^{n-1}[(m+n)c-(mb+na)]=0, \text{ from (1)}$$

$$\Rightarrow (m+n)c-(mb+na)=0, \because c\neq a, c\neq b$$

$\Rightarrow c=\dfrac{mb+na}{m+n} \in (a,b)$, *as* this is the point in (a, b) which divides a and b in the ratio $m{:}n$ internally

$\therefore$ Rolle's theorem *is* verified

Prob : *Verify* Rolle's theorem for the function $f(x) = 2 + (x-1)^{\frac{2}{3}}$ *in* the interval $[0,2]$

Sol : *Given* function $f(x) = 2 + (x-1)^{\frac{2}{3}}$ *in* the interval $[0,2]$

$$\text{Now } f'(x) = 0 + \frac{2}{3}(x-1)^{\frac{2}{3}-1}$$

$$= \frac{2}{3}(x-1)^{-\frac{1}{3}}$$

Which does not exist when $x = 1 \in (0,2)$

$\therefore$ f is not derivable on $(0,2)$

$\therefore$ Rolle's theorem is not applicable to the given function.

$Prob : Verify$ Rolle's theorem for the function $f(x) = \frac{Sinx}{e^x}$ on $[0, \pi]$

$Sol : Given$ function is $f(x) = \frac{Sinx}{e^x}, \forall x \in [0, \pi]$

$$Also\ f^{'}(x) = \frac{e^x Cosx - Sinx(e^x)}{e^x}$$

$$\Rightarrow f^{'}(x) = Cosx - Sinx, e^x \neq 0, \forall x \in R \text{ ----(1)}$$

clearly this exists $\forall$ x $\in$ R

$\therefore f^{'}(x)$ exists i.e f is derivable on $[0, \pi]$

Since derivability imply continuity, f is continuous on $[0, \pi]$

Also $f(0) = 0$ & $f(\pi) = 0$ so that $f(0) = f(\pi)$

$\therefore f$ satisfies all the conditions of Rolle' s theorem.

$\therefore$ By Rolle' s theorem $\exists$ c $\in$ (a, b) so that $f^{'}(c) = 0$

$$\Rightarrow Cosc - Sinc = 0, \text{ from (1)}$$

$$\Rightarrow Tanc = 1$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

$\therefore$ Rolle' s theorem *is* verified

$\text{Prob}: \textit{Verify}$ Lagrange's theorem for the function $f(x) = x(x-1)(x-2)$ on $[0, \frac{1}{2}]$

$\textit{Sol}: \textit{Given}$ function is $f(x) = x(x-1)(x-2)$ on $[0, \frac{1}{2}]$

$$\Rightarrow f(x) = x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$$

Being a polynomial functio $f(x)$ *is* continuous on R i.e on $[0, \frac{1}{2}]$

Also $f'(x) = 3x^2 - 6x + 2$, *exists* $\forall$ x $\in$ R i.e on $[0, \frac{1}{2}]$

$\therefore f'(x)$ exists i.e f is derivable on $[0, \frac{1}{2}]$

$\therefore f$ satisfies the two conditions of Lagrange's Mean value theorem.

$\therefore$ By Lagrange's Mean value theorem $\exists$ c $\in$ (a, b) so that $f'(c) = \dfrac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$ ---(1)

$$\text{Now} \quad f(\tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2) = \tfrac{1}{2} \times -\tfrac{1}{2} \times -\tfrac{3}{2} = \tfrac{3}{8}$$

$$\& \qquad f(0) = 0$$

$$\textit{Now}(1) \Rightarrow f'(c) = \frac{\frac{3}{8} - 0}{\frac{1}{2}} = \frac{3}{4}$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

Since derivability imply continuity, f is continuous on $[0, \pi]$

$\therefore$ Rolle's theorem *is* verified

$$Now(1) \Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$\Rightarrow c = \frac{-(-24) \pm \sqrt{576 - 240}}{24}$$

$$= \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm \sqrt{16 \times 21}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24}$$

$$= \frac{6 \pm \sqrt{21}}{6}$$

$$\therefore \text{The two values of c are } \frac{6+\sqrt{21}}{6}, \frac{6-\sqrt{21}}{6}$$

$$But \text{ out of these } \frac{6+\sqrt{21}}{6} \notin (0, \frac{1}{2})$$

$$\therefore \text{The possible value of c is c} = \frac{6-\sqrt{21}}{6} \in (0, \frac{1}{2})$$

$$\therefore \text{ Lagrange's mean value theorem } is \text{ verified}$$

$\Pr ob : Verify$ Lagrange's theorem for the function $f(x) = lx^2 + mx + n , x \in [a, b]$

$Sol : Given$ function is $f(x) = lx^2 + mx + n , x \in [a, b]$

$Being$ a polynomial functio $f(x) is$ continuous on R i.e on [a, b]

$Also \ f'(x) = 2lx + m, exists \ \forall \ x \in R$ i.e on [a, b]

$\therefore \ f'(x)$ exists i.e f is derivable on $[a, b]$

$\therefore \ f$ satisfies the two conditions of Lagrange's Mean value theorem.

$\therefore$ By Lagrange's Mean value theorem $\exists \ c \in (a, b)$ so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ ---(1)

$$Now \quad f(a) = la^2 + ma + n$$

$$\& \quad f(b) = lb^2 + mb + n$$

$$Now \ f(b) - f(a) = \frac{lb^2 + mb + n - la^2 - ma - n}{b\text{-}a}$$

$$= \frac{l(b^2 - a^2) + m(b - a)}{b\text{-}a}$$

$$= l(b + a) + m$$

$$\therefore (1) \Rightarrow 2lc + m = l(b + a) + m$$

$$\Rightarrow 2lc = l(b + a)$$

$\Rightarrow c = \dfrac{a + b}{2} \in (a, b), as$ this is arithmetic mean of $a \ and \ b$

$\therefore$ Lagrange's Mean value theorem is verified

$\text{Prob}: \textit{Find}$ c of Caucy's mean value theorem for the functions

$$f(x)=\sqrt{x}\,,\ g(x)=\frac{1}{\sqrt{x}}\,,\ x\in[a,b]$$

$\textit{Sol}:\textit{Given}$ functions are $f(x)=\sqrt{x}\,,\ g(x)=\frac{1}{\sqrt{x}}\,,\ x\in[a,b]$

Then $f'(x)=\frac{1}{2\sqrt{x}}\,,\ g'(x)=-\frac{1}{2}x^{-\frac{1}{2}-1}=-\frac{1}{2x^{\frac{3}{2}}}$

$\textit{Also } f'(x)\ \&\ g'(x) \textit{ exists } \forall\ x\in[a,b]$

$\therefore$ By Cauchy's Mean value theorem $\exists\, c\in(a,b)$ so that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c\sqrt{c}}}=\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\Rightarrow -c=\frac{\sqrt{b}-\sqrt{a}}{\left[\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}\sqrt{b}}\right]}$$

$$\Rightarrow c=\sqrt{ab}$$

Clearly $c=\sqrt{ab}\in(a,b)$.